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GEOMETRY.

215. Proposed by M. J. NEWELL, A. M., Evanston High School, Evanston, Ill.

Construct geometrically a right triangle, given the bisectors of the acute angles.

Solution by L. E. DICKSON, Ph. D., The University of Chicago.

In the absence of a purely geometrical construction, I show that the triangle is uniquely determined, a leg a being given by a quartic equation having a single positive root. Denote the acute angles of the right triangle by 2α and 2β , their bisectors by Q and P , the opposite sides by a and b , the hypotenuse by h . Then $\alpha + \beta = 45^\circ$. By a familiar theorem on bisectors,

$$P^2 = 2a^2 h \div (a + h) \dots\dots\dots (1).$$

$$\therefore h = Pa^2 \div (2a^2 - P^2) \dots\dots\dots (2).$$

Now $b = Q \cos \alpha = Q \cos(45^\circ - \beta) = Q(\cos \beta + \sin \beta) \div \sqrt{2}$. Hence

$$b\sqrt{2} = Q \left(\frac{a}{P} + \frac{ab}{P(a+h)} \right), \text{ or } b \left(\frac{P}{Q} \sqrt{2} - \frac{a}{a+h} \right) = a \dots\dots\dots (3).$$

Substituting in (3) the value of h from (2), we get

$$b \left(\frac{P}{Q} \sqrt{2} - \frac{2a^2 - P^2}{2a^2} \right) = a \dots\dots\dots (4).$$

Similarly,

$$a \left(\frac{Q}{P} \sqrt{2} - \frac{2b^2 - Q^2}{2b^2} \right) = b \dots\dots\dots (5).$$

Eliminating b between (4) and (5), we get

$$\left[\frac{P}{Q} \sqrt{2} - 1 + \frac{P^2}{2a^2} \right] \left[\frac{Q}{P} \sqrt{2} - 1 + \frac{Q^2}{2a^2} \left(\frac{P}{Q} \sqrt{2} - 1 + \frac{P^2}{2a^2} \right)^2 \right] = 1.$$

Multiplying by $16a^8 PQ$ and expanding, we obtain

$$\begin{aligned} & -16\sqrt{2} (P^2 + Q^2 - \sqrt{2} PQ) a^8 + (16\sqrt{2} P^4 - 48P^3 Q + 32\sqrt{2} P^2 Q^2 - 16Q^3 P) a^6 \\ & + 3(2\sqrt{2} P - 2Q)^2 P^3 Q a^4 + 3a^2 (2\sqrt{2} P - 2Q) P^5 Q^2 + P^7 Q^3 = 0 \dots\dots\dots (6). \end{aligned}$$

[As a check we note that for $a=b$, $P^2=Q^2=2(2-\sqrt{2})a^2$ and (6) is satisfied]. The coefficient of a^6 equals $16P(\sqrt{2} P - Q)(P^2 + Q^2 - \sqrt{2} PQ)$. Set $\kappa = 16(P^2 + Q^2 - \sqrt{2} PQ)$ and $\lambda = (\sqrt{2} P - Q)P$. Then (6) becomes

$$a^8 - \frac{1}{2}\lambda\sqrt{2} a^6 - 6\sqrt{2} \kappa^{-1}\lambda^2 PQ a^4 - 3\sqrt{2}\kappa^{-1}\lambda P^4 Q^2 a^2 - \frac{1}{2}\sqrt{2} \kappa^{-1} P^7 Q^3 = 0 \dots\dots\dots (7).$$

Let $P \geq Q$, so that λ and κ are positive. There being a single change of sign in (7), there is one and but one positive root by Descartes' Rule of Signs.

A more elementary but less fortunate method consists in using (1) and the corresponding relation $Q^2=2hb^2 \div (b+h)$ (8).

Now from (1), $h(2a^2-P^2)=P^2a$. But $h^2=a^2+b^2$. Hence

$$b^2=4a^4(P^2-a^2) \div (2a^2-P^2)^2 \text{ (9).}$$

Eliminating h between (2) and (8), we get

$$P^4a^2(2b^2-Q^2)^2=Q^4b^2(2a^2-P^2)^2.$$

In this we substitute the value of b^2 from (9) and obtain an equation of the sixth degree for a^2 . Set $a=2a^2-P^2$. Then

$$(P^4+Q^4)a^6+2P^4(P^2+Q^2)a^5+P^6(2Q^2-P^2)a^4-2P^3(2P^2+Q^2)a^3 \\ -P^{10}(2Q^2+P^2)a^2+2P^{14}a+P^{16}=0 \text{ (10).}$$

We may take $P \leq Q$. Then by Descartes' Rule of Signs, there are two or no positive roots. There are two positive roots, so that (10) does not uniquely determine the leg a .

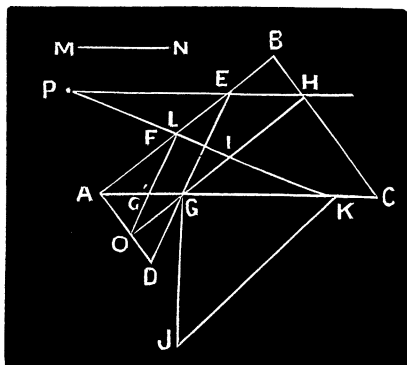
218. Proposed by O. W. ANTHONY, DeWitt Clinton High School, New York City.

From a given triangle cut off an area equivalent to a given square by a line passing through a given point without the triangle.

IV. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC be the given triangle, MN a side of the given square, P the given point. Through P draw PE parallel to AC meeting AB in E . Perpendicular to AB draw $AD=MN$, lay off $AF=MN$. Join DE and draw FO parallel to DE , cutting AC in G' . Draw OGH parallel to AB . At G erect $GJ=PE$ perpendicular to AC and draw $JK=PH$. Draw $PLIK$.

From similar triangles AED and AFO , we have $AE:AD=AF(=AD):AO$.
 $\therefore AD^2 = AE \times AO = \text{area } AEHG$; $JK^2 - JG^2 = GK^2$ or $PHJ - PEL = GIK = LEHI$.
 $\therefore ALM = MN^2$.



V. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let ABC be the triangle and P the given point without. Draw PE parallel to AB , cutting AC in D . Make parallelogram $DEFA$ =given square. On F erect the perpendicular $FG=PD$, and make $GQ=PE$. Connect P with Q , then will PQ be the required line.